Exponents and Radicals

$$\sqrt{(a+b)}^{10}$$

Exponents are a very important part of algebra. An exponent is just a convenient way of writing repeated multiplications of the same number.

Radicals involve the use of the radical sign (√). Sometimes these are called **surds**.

If you learn the rules for exponents and radicals, then your enjoyment of mathematics will surely increase!

1.- Objetives

Unit Objectives

- To be able to illustrate the relationship between the radical and exponential forms of an equation. Supported by learning objectives 1, 3, and 5.
- To demonstrate the ability to work with operations involving radical numbers. Supported by learning objectives 2, 4, 6, and 7.
- To be able to solve equations involving radicals and to be able to justify the solutions. Supported by learning objectives 8 to 11.

Learning Objectives

1. To evaluate powers with rational exponents.

2. To apply the laws of exponents to simplify expressions involving rational exponents.

3. To write exponential expressions in radical form.

4. To simplify square root and cube root expressions.

5. To write radical expressions in exponential form.

6. To add, subtract, multiply and divide square root and cube root expressions.

7. To rationalize monomial and binomial denominators in radical expressions.

8. To solve and verify the solutions of quadratic equations by factoring, completing the trinomial square, and using the quadratic formula.

9. To solve and verify equations involving absolute value.

10. To solve radical equations with two unlike radicands.

11. To solve word problems involving radical equations.

2.-Contents

2.1. Simplifying Expressions with Integral Exponents

Laws of Exponents

Integral Exponents

Integral here means "integer". So the exponent (or power) is an integer. [That is, either a negative whole number, 0, or a positive whole number.]

Definition: a^m means "multiply a by itself m times". That is:

$a^m = a \times a \times a \times a \times a \times ... \times a$
[We do the multiplication m times.]

Note: This definition only really holds for $m > 0$, since it doesn't make a lot of sense if m is negative. (You can't multiply something by itself -3 times! And how does multiplying something by itself 0 times give 1?)

In such cases we have to rely on patterns and conventions to define what is going on. See below for zero and negative exponents.

Examples

(1) $y^5 = y \times y \times y \times y \times y$

(2) $2^4 = 2 \times 2 \times 2 \times 2 = 16$.

Multiplying Expressions with the Same Base

Definition: $a^m \times a^n = a^{m+n}$

Let's see how this works with an example.

Example

$b^5 \times b^3 = (b \times b \times b \times b \times b) \times (b \times b \times b)$

$= b^8$ (that is, b^{5+3})

Dividing Expressions with the Same Base

Definition: $\dfrac{a^m}{a^n} = a^{m-n}$ (Of course, $a \neq 0$)

It may be easiest to see how this one works with an example.

Example

$$\frac{b^7}{b^2} = \frac{b \times b \times b \times b \times b \times b \times b}{b \times b} = b^5$$

We cancel 2 of the *b*'s from the numerator and the two *b*'s from the denominator of the fraction. The result is equivalent to b^{7-2}.

Repeated Multiplication of a Number Raised to a Power

$(a^m)^n = (a^m) \times (a^m) \times (a^m) \times ... \times (a^m)$ [We multiply *n* times].

$= a^{mn}$

So we write:

Definition: $(a^m)^n = a^{mn}$

Example

$(p^3)^2 = p^3 \times p^3$

$= (p \times p \times p) \times (p \times p \times p)$

$= p^6$

A Product Raised to an Integral Power

Definition: $(ab)^n = a^n b^n$

Example

$(5q)^3 = 5^3q^3 = 125q^3$

A Fraction Raised to an Integral Power

Definition: $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

Example

$$\begin{aligned}\left(\frac{2}{3}\right)^4 &= \left(\frac{2}{3}\right) \times \left(\frac{2}{3}\right) \times \left(\frac{2}{3}\right) \times \left(\frac{2}{3}\right) \\ &= \frac{2 \times 2 \times 2 \times 2}{3 \times 3 \times 3 \times 3} \\ &= \frac{2^4}{3^4} \\ &= \frac{16}{81}\end{aligned}$$

Zero Exponents

Definition: $a^0 = 1$ ($a \neq 0$)

Example

$7^0 = 1$

Note: $a^0 = 1$ is a **convention**, that is, we agree that raising any number to the power 0 is 1. We cannot multiply a number by itself zero times.

In the case of zero raised to the power 0 (0^0), mathematicians have been debating this for hundreds of years. It is most commonly regarded as having value 1, but is not so in all places where it occurs. That's why we write $a \neq 0$.

Negative Exponents

Definition: $a^{-n} = \frac{1}{a^n}$ (Once again, $a \neq 0$)

In this exponent rule, a cannot equal 0 because you cannot have 0 on the bottom of a fraction.

Example

$$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$$

Explanations

Observe the following decreasing pattern:

$3^4 = 81$

$3^3 = 27$

$3^2 = 9$

$3^1 = 3$

For each step, we are dividing by 3. Now, continuing beyond 3^1 and dividing by 3 each times gives us:

$$3^0 = \frac{3}{3} = 1$$

$$3^{-1} = \frac{1}{3}$$

$$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$$

$$3^{-3} = \frac{1}{3^3} = \frac{1}{27}$$

$$3^{-4} = \frac{1}{3^4} = \frac{1}{81}$$

Summary - Laws of Exponents

$$a^m \times a^n = a^{m+n}$$

$$\frac{a^m}{a^n} = a^{m-n} \quad (a \neq 0)$$

$$(a^m)^n = a^{mn}$$

$$(ab)^n = a^n b^n$$

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

$$a^0 = 1 \qquad (a \neq 0)$$

$$a^{-n} = \frac{1}{a^n} \qquad (a \neq 0)$$

[**Note:** These laws mostly apply if we have fractional exponents, which we meet in the next section, Fractional Exponents.]

Let's try some mixed examples where we have integral exponents.

Example (1)

(a) Simplify $a^5 \times a^{-3}$

(b) Simplify $a^3 \times a^{-5}$

Example (2)

Simplify $(2^3 \times 2^{-4})^2$

Example (3)

Simplify $\dfrac{a^2b^3c^0}{ab^7}$

Note the following differences carefully:

$(-5x)^0 = 1$, but $-5x^0 = -5$.

Similarly:

$(-5)^0 = 1$, but $-5^0 = -1$.

Example (4)

Simplify $(2a + b^{-1})^{-2}$

Exercises

Q1 $(5an^{-2})^{-1}$

Q2 $\left(\dfrac{a^{-2}}{b^2}\right)^{-3}\left(\dfrac{a^{-3}}{b^5}\right)^{2}$

Q3 $(2a - b^{-2})^{-1}$

2.2 Fractional Exponents

Fractional exponents can be used instead of using the radical sign (√). We use fractional exponents because often they are more convenient, and it can make algebraic operations easier to follow.

Fractional Exponent Laws

The ***n*-th root** of a number can be written using the power $1/n$, as follows:

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

Meaning: The n-th root of a when multiplied n times, gives us a.

$a^{1/n} \times a^{1/n} \times a^{1/n} \times \ldots \times a^{1/n} = a$

Definitions: The number under the radical is called the *radicand* (in the above case, the number a), and the number indicating the root being taken is called the *order* (or *index*) of the radical (in our case n).

Example

The 4-th root of 625 can be written as either:

$625^{1/4}$

or equivalently, as

$$\sqrt[4]{625}$$

Its value is 5, since $5^4 = 625$.

Raising the *n*-th root to the Power *m*

If we need to raise the n-th root of a number to the power m (say), we can write this as:

$$a^{\frac{m}{n}} = (\sqrt[n]{a})^m$$

In English, this means "take the n-th root of the number, then raise the result to the power m".

Example 1

Simplify $8^{\frac{2}{3}}$

Example 2

(a) Simplify $\left(8a^2b^4\right)^{\frac{1}{3}}$

(b) Simplify $a^{\frac{3}{4}}a^{\frac{4}{5}}$

Example (3)

Simplify

$$\left(\frac{4^{-\frac{3}{2}}x^{\frac{2}{3}}y^{-\frac{7}{4}}}{2^{\frac{3}{2}}x^{-\frac{1}{3}}y^{\frac{3}{4}}}\right)^{\frac{2}{3}}$$

Exercises

Q1 $5^{1/2}5^{3/2}$

Q2 $\dfrac{1000^{1/3}}{400^{-1/2}}$

2.3 Simplest Radical Form

Before we can simplify radicals, we need to know some rules about them. These rules just follow on from what we learned in the first 2 sections in this chapter, Integral Exponents and Fractional Exponents.

Expressing in **simplest radical form** just means simplifying a radical so that there are no more square roots, cube roots, 4th roots, etc left to find. It also means removing any radicals in the denominator of a fraction.

Laws of Radicals

n-th root of a Number to the Power *n*

We met this idea in the last section, Fractional Exponents. Basically, finding the *n*-th root of a number is the opposite of raising the number to the power *n*, so they effectively cancel each other out. These 4 expressions have the same value:

$$\sqrt[n]{a^n} = \left(\sqrt{a^n}\right)^n = \left(a^{1/n}\right)^n = a$$

For the simple case, these all have the same value:

$\sqrt{a}^2 = (\sqrt{a})^2 = \sqrt{(a^2)} = a$

The Product of the *n*-th root of *a* and the *n*-th root of *b* is the *n*-th root of *ab*

$$\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$$

We could write this using fractional exponents as well:

$$a^{1/n} \times b^{1/n} = (ab)^{1/n}$$

The *m*-th Root of the *n*-th Root of the Number *a* is the *mn*-th Root of *a*

$$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$$

The equivalent expression using fractional exponents is as follows:

$$\left(a^{1/n}\right)^{1/m} = a^{1/mn}$$

The *n*-th Root of *a* Over the *n*-th Root of *b* is the *n*-th Root of *a/b*

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}} \quad (b \neq 0)$$

If we write the same thing using fractional exponents, we have:

$$\frac{a^{1/n}}{b^{1/n}} = \left(\frac{a}{b}\right)^{1/n} \quad (b \neq 0)$$

Example 1.

Simplify the following:

(a) $\sqrt[5]{4^5}$

Answer:

$$\sqrt[5]{4^5} = \left(\sqrt[5]{4}\right)^5 = 4$$

We have used the first law above.

(b) $\sqrt[3]{2}\sqrt[3]{3}$

Answer:

$$\sqrt[3]{2}\sqrt[3]{3} = \sqrt[3]{2 \times 3} = \sqrt[3]{6}$$

We have used $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$.

(c) $\sqrt[3]{\sqrt{5}}$

Answer:

$$\sqrt[3]{\sqrt{5}} = \sqrt[3\times2]{5} = \sqrt[6]{5}$$

We have used the law: $\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$

(d) $\dfrac{\sqrt{7}}{\sqrt{3}}$

Answer:

$$\frac{\sqrt{7}}{\sqrt{3}} = \sqrt{\frac{7}{3}}$$

Nothing much to do here. We used: $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\dfrac{a}{b}}$

Example 2

In these examples, we are expressing the answers in simplest radical form, using the laws given above.

(a) $\sqrt{72}$

Answer:

We need to examine 72 and find the highest square number that divides into 72. (Squares are the numbers $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, $4^2 = 16$, ...)

In this case, 36 is the highest square that divides into 72 evenly. We express 72 as 36 × 2 and proceed as follows.

$$\sqrt{72} = \sqrt{(36)(2)} = \sqrt{36}\sqrt{2} = 6\sqrt{2}$$

We have used the law: $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$

(b) $\sqrt{a^3b^2}$

Answer:

$$\sqrt{a^3b^2} = \sqrt{(a^2)(a)(b^2)} = \sqrt{a^2}\sqrt{a}\sqrt{b^2} = ab\sqrt{a}$$

We have used the law: $\sqrt{a^2} = a$

(c) $\sqrt[3]{40}$

Answer:

$$\sqrt[3]{40} = \sqrt[3]{8 \times 5} = \sqrt[3]{8}\sqrt[3]{5} = 2\sqrt[3]{5}$$

(d) $\sqrt[5]{64x^8y^{12}}$

Answer:

$$\begin{aligned}\sqrt[5]{64x^8y^{12}} &= \sqrt[5]{(32)(2)(x^5)(x^3)(y^{10})(y^2)}\\ &= \sqrt[5]{32(x^5)(y^{10})}\sqrt[5]{2x^3y^2}\\ &= 2xy^2\sqrt[5]{2x^3y^2}\end{aligned}$$

Exercises. Simplify:

Q1 $\sqrt{12ab^2}$

Q2 $\sqrt[4]{64r^3s^4t^5}$

Q3 $\sqrt{\frac{x}{2x+1}}$

This one requires a special trick. To remove the radical in the denominator, we need to multiply top and bottom of the fraction by the denominator.

2.4 Addition and Subtraction of Radicals

In algebra, we can combine terms that are *similar* eg.

$2a + 3a = 5a$

$8x^2 + 2x - 3x^2 = 5x^2 + 2x$

Similarly for surds, we can combine those that are similar. They must have the same radicand (number under the radical) and the same index (the root that we are taking).

Example 1

(a) $2\sqrt{7} - 5\sqrt{7} + \sqrt{7}$

Answer:

$$2\sqrt{7} - 5\sqrt{7} + \sqrt{7} = -2\sqrt{7}$$

We can do this because the radicand is 7 in each case and the index is 1/2 (that is, we are taking square root) in each case.

(b) $\sqrt[5]{6} + 4\sqrt[5]{6} - 2\sqrt[5]{6}$

Answer:

$$\sqrt[5]{6} + 4\sqrt[5]{6} - 2\sqrt[5]{6} = 3\sqrt[5]{6}$$

(c) $\sqrt{5} + 2\sqrt{3} - 5\sqrt{5}$

Answer:

$$\sqrt{5}+2\sqrt{3}-5\sqrt{5}=2\sqrt{3}-4\sqrt{5}$$

Example 2

(a) $6\sqrt{7}-\sqrt{28}+3\sqrt{63}$

Answer:

In each part, we are taking square root, but the number under the square root is different. We need to simplify the radicals first and see if we can combine them.

$$\begin{aligned} 6\sqrt{7}-\sqrt{28}+3\sqrt{63} &= 6\sqrt{7}-\sqrt{4\times 7}+3\sqrt{9\times 7} \\ &= 6\sqrt{7}-2\sqrt{7}+3(3\sqrt{7}) \\ &= 6\sqrt{7}-2\sqrt{7}+9\sqrt{7} \\ &= 13\sqrt{7} \end{aligned}$$

(b) $3\sqrt{125}-\sqrt{20}+\sqrt{27}$

Answer:

$$\begin{aligned} 3\sqrt{125}-\sqrt{20}+\sqrt{27} &= 3\sqrt{25\times 5}-\sqrt{4\times 5}+\sqrt{9\times 3} \\ &= 3(5\sqrt{5})-2\sqrt{5}+3\sqrt{3} \\ &= 13\sqrt{5}+3\sqrt{3} \end{aligned}$$

Example 3

Simplify:

$$\sqrt{\frac{2}{3a}}-2\sqrt{\frac{3}{2a}}$$

Answer:

Our aim here is to remove the radicals from the denominator of each fraction and then to combine the terms into one expression.

$$\sqrt{\frac{2}{3a}} - 2\sqrt{\frac{3}{2a}} = \sqrt{\frac{2(3a)}{3a(3a)}} - 2\sqrt{\frac{3(2a)}{2a(2a)}}$$

$$= \sqrt{\frac{6a}{9a^2}} - 2\sqrt{\frac{6a}{4a^2}}$$

$$= \frac{1}{3a}\sqrt{6a} - \frac{2}{2a}\sqrt{6a}$$

$$= \frac{1}{3a}\sqrt{6a} - \frac{1}{a}\sqrt{6a}$$

$$= \frac{\sqrt{6a} - 3\sqrt{6a}}{3a}$$

$$= \frac{-2\sqrt{6a}}{3a}$$

$$= -\frac{2}{3a}\sqrt{6a}$$

We multiply top and bottom of each fraction with their denominators. This gives us a perfect square in the denominator in eachcase, and we can remove the radical.

We then simplify and see that we have like terms ($\sqrt{6a}$).

We then proceed to subtract the fractions by finding a common denominator ($3a$).

Exercises

Q1) $\sqrt{7} + \sqrt{63}$

Q2) $2\sqrt{44} - \sqrt{99} + \sqrt{2}\sqrt{88}$

Q3) $\sqrt[6]{\sqrt{2}} - \sqrt[12]{2^{13}}$

2.5 Multiplication and Division of Radicals

When multiplying expressions containing radicals, we use the law $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$, along with normal procedures of algebraic multiplication.

Example 1

(a) $\sqrt{5}\sqrt{2}$

Answer:

$$\sqrt{5}\sqrt{2} = \sqrt{5 \times 2} = \sqrt{10}$$

(b) $\sqrt{33}\sqrt{3}$

Answer:

$$\sqrt{33}\sqrt{3} = \sqrt{99} = \sqrt{9 \times 11} = 3\sqrt{11}$$

Example 2

(a) $\sqrt[3]{6}\sqrt[3]{4}$

Answer:

$$\sqrt[3]{6}\sqrt[3]{4} = \sqrt[3]{6 \times 4} = \sqrt[3]{24} = \sqrt[3]{8 \times 3} = \sqrt[3]{8}\sqrt[3]{3} = 2\sqrt[3]{3}$$

In this case, we needed to find the largest cube that divides into 24. (The cubes are the numbers $1^3 = 1$, $2^3 = 8$, $3^3 = 27$, $4^3 = 64$, ...)

(b) $\sqrt[5]{8a^3b^4}\sqrt[5]{8a^2b^3}$

Answer:

$$\begin{aligned}\sqrt[5]{8a^3b^4}\sqrt[5]{8a^2b^3} &= \sqrt[5]{(8a^3b^4)(8a^2b^3)}\\ &= \sqrt[5]{64a^5b^7}\\ &= \sqrt[5]{32a^5b^5}\sqrt[5]{2b^2}\\ &= 2ab\sqrt[5]{2b^2}\end{aligned}$$

Example 3

(a) $(3+\sqrt{5})^2$

Answer:

$$\begin{aligned}(3+\sqrt{5})^2 &= 3^2 + 2(3)(\sqrt{5}) + (\sqrt{5})^2\\ &= 9 + 6\sqrt{5} + 5\\ &= 14 + 6\sqrt{5}\end{aligned}$$

(b) $(\sqrt{a}-\sqrt{b})^2$

Answer:

$$\begin{aligned}(\sqrt{a}-\sqrt{b})^2 &= (\sqrt{a}\ )^2 - 2(\sqrt{a})(\sqrt{b}) + (\sqrt{b})^2\\ &= a + b - 2\sqrt{ab}\end{aligned}$$

Division of Radicals (Rationalizing the Denominator)

This process is also called "rationalising the denominator" since we remove all irrational numbers in the denominator of the fraction.

This is important later when we come across Complex Numbers.

Reminder: From earlier algebra, you will recall the difference of squares formula:

$(a + b)(a - b) = a^2 - b^2$

We will use this formula to rationalize denominators.

Example

Simplify: $\dfrac{1}{\sqrt{3} - \sqrt{2}}$

Answer

The question requires us to divide 1 by ($\sqrt{3} - \sqrt{2}$).

We need to multiply top and bottom of the fraction by the **conjugate** of ($\sqrt{3} - \sqrt{2}$).

The conjugate is easily found by reversing the sign in the middle of the radical expression. In this case, our minus becomes plus. So the conjugate of ($\sqrt{3} - \sqrt{2}$) is ($\sqrt{3} + \sqrt{2}$).

$$\begin{aligned}\frac{1}{\sqrt{3} - \sqrt{2}} &= \frac{1}{\sqrt{3} - \sqrt{2}} \times \frac{\sqrt{3} + \sqrt{2}}{\sqrt{3} + \sqrt{2}} \\ &= \frac{\sqrt{3} + \sqrt{2}}{(\sqrt{3})^2 - (\sqrt{2})^2} \\ &= \frac{\sqrt{3} + \sqrt{2}}{3 - 2} \\ &= \sqrt{3} + \sqrt{2}\end{aligned}$$

After we multiply top and bottom by the conjugate, we see that the denominator becomes free of radicals (in this case, the denominator has value 1).

Historical Note

In the days before calculators, it was important to be able to rationalize denominators. Using logarithm tables, it was very troublesome to find the value of expressions like our example above.

Now that we use calculators, it is not so important to rationalize denominators.

However, rationalizing denominators is still used for several of our algebraic techniques (see especially Complex Numbers), and is still worth learning.

Exercises

Q1) $\sqrt[5]{4}\sqrt[5]{16}$

Q2 **)** $\sqrt{a}(\sqrt{ab} + \sqrt{c^3})$

Q3) $\dfrac{2\sqrt{x}}{\sqrt{x} - \sqrt{y}}$

Q4) $\dfrac{6\sqrt{a}}{2\sqrt{a} - b}$

2.6 Equations With Radicals

It is important in this section to **check your solutions** in the original equation, as we often produce solutions which actually don't work when substituted back into the original equation.

(In fact, it is **always** good to check solutions for equations - you learn so much more about why things are working the way they do.)

Example:

Solve $\sqrt{2x + 6} = 2x$

Answer

Squaring both sides gives:

$$2x + 6 = 4x^2$$
$$4x^2 - 2x - 6 = 0$$
$$2x^2 - x - 3 = 0$$
$$(2x - 3)(x + 1) = 0$$

So our solutions are x = 1.5 or -1.

CHECK:

Substituting x = 1.5 in our original equation gives:

$$LHS = \sqrt{2(1.5) + 6} = \sqrt{9} = 3$$
$$RHS = 2 \times 1.5 = 3$$

Checks OK

Substituting x = -1 in our original equation gives:

$$LHS = \sqrt{2(-1) + 6} = \sqrt{4} = 2$$
$$RHS = 2 \times -1 = -2$$

DOES NOT check OK

So we conclude the only solution is x = 1.5.

Can you figure out **where** the 'wrong' solution is coming from?

Exercises

1. Solve $\sqrt{3x + 4} = x$

2. Solve $\sqrt{13 + \sqrt{x}} = \sqrt{x} + 1$

3. Solve $\sqrt[4]{x + 10} = \sqrt{x - 2}$

4. In the study of spur gears in contact, the equation

$$kC = \sqrt{R_1^2 - R_2^2} + \sqrt{r_1^2 - r_2^2} - A$$

is used. Solve for r_1^2.

3.- Procedures and attitudes

1.- Students should be asked to recall the basic laws of exponents and to demonstrate their understanding of integral exponents. This could be done as a class activity. These could be listed as they are completed, so all students will have the opportunity to recall the basic facts.

The students may be asked to consider the likely values of an expression of the type $25^{1/2}$. Students could discuss possible answers in small groups and be instructed to provide a rationale for their group answer.

If no group seems to approach the solution, the teacher could provide a series of clues for the students to utilize, beginning with the properties of exponents.

E.g.: $25^{1/2} = (?)^{1/2} = (5^2)^{1/2} = (5)^{2(1/2)}$

$= 5^1 = 5$

In the above example, the students should develop, or be shown, the relationship of an exponent of 1/2 and the square root. The notation for a rational exponent can be introduced,

$(x^{1/m} = \sqrt[m]{x^1})$,

and several examples done by the teacher, class, or individual students. The definition can be extended to

$(X^{n/m} = \sqrt[m]{X^n})$

and more examples considered.

2.- Some statements could be given to the students to work on in pairs or in small groups. These should give the students some practice in working with rational exponents.

E.g:

a) $x^{2/3} \cdot x^{5/3}$

b) $(x^{1/2})^{2/3}$

Once the students have had practice in working with the laws of exponents, this can be extended to simplification, or evaluation exercises as well.

E.g.: $x^{1/2} \cdot x^{1/3}$

$8^{1/2} \cdot 8^{1/6}$

3.- Students will be expected to be able to demonstrate that they understand the definition of a rational exponent by writing a given radical expression in radical form. Students should be instructed to review their definition of a rational exponent and to apply that definition to a set of exercises in which they would practise this concept.

They may work individually, in pairs, small groups, or as an entire class, in practising this concept.

4.- A review of the simplifying process of square roots as studied in Mathematics 20 may be a useful starting point.

When the students have recalled the principles of simplifying square root radicals, they can be introduced to cube root simplification.

It may be necessary to have the students compare the square root and the cube root of various numbers, to help them discover that the cube root of a negative number does in fact exist. This can be tied to the definition of a rational exponent and aid understanding of the role of an even or odd denominator in rational exponents.

There are several different methods by which this concept can be explained; check your reference texts for different explanations. Some students may prefer to look at some of these options.

Note that one common error that students tend to make with $\sqrt{\ }$ is

$$\sqrt{(a + b)} = \sqrt{a} + \sqrt{b}$$

Try some numerical examples to show this is not the case.

Ask students if there are any cases where

$$\sqrt{(a + b)} = \sqrt{a} + \sqrt{b}?$$

5.- Students are expected to become familiar with converting from exponential expressions to radical form and vice versa. In objective D.3, the former was introduced and practice given.

The teacher may find it useful to combine objectives D.3 and D.5 in one lesson.

In any case, some practice should be given to the students to work on converting from radical to exponential form. The students could practise individually, in pairs, or in small groups.

6.- Square root operations were introduced in third year. Students may be asked to recall the basic procedures involved in these operations. Students can be given a few exercises to do to review these third year operations.

The introduction of cube root operations can be done as an entire class or by getting individual or group input when developing these procedures.

Students could be given a set of exercises to use in developing their skills in dealing with square and cube root operations. Students could work in pairs or in small groups in doing these exercises.

7.- Students can be introduced to the process of rationalizing by practicing selected exercises, and being asked to examine the results obtained in these situations.

E.g.:
$\sqrt{5} * \sqrt{5} =$
$3\sqrt{7} * 3\sqrt{7} * 3\sqrt{7} =$
$(3\sqrt{2} - \sqrt{5})(3\sqrt{2} - \sqrt{5}) =$

They can be asked to determine an expression that could be multiplied by a given expression so that the result is a rational number.

E.g.:
$(4\sqrt{5} - 3\sqrt{7})(?)$

a rational result.

Once the basic process of converting an irrational denominator to a rational denominator is observed by the students, they could be introduced to exercises which involve rationalizing the denominator. They could work on exercises individually, in pairs, in small groups, or as a class.

Students should be expected to summarize their results, and describe the processes utilized for denominators with square roots, and for denominators with cube roots. The teacher may wish to have students theorize about a process that could be utilized for denominators with other indices.

8.- Students have already solved quadratic equations by factoring, and by taking the square root of both sides of the equation, in third year. As well, they have already reviewed these types previously in fourth year.

Students should be given some practice in factoring perfect trinomial squares, and in determining the value of c in $ax^2+ bx+c$, which would make ax^2+bx+c a perfect square. They could work cooperatively on these skills. Students should be able to summarize their findings, draw conclusions about the completion of a perfect square, and demonstrate their understanding by completing other examples.

Once students are able to complete the square, this process can be utilized in solving quadratic equations. Students should observe some examples and work cooperatively to solve (and check) several examples of their own.

The quadratic formula can then be developed as an exercise in solving quadratic equations by completing the square for the general case $ax^2 +bx+c$. Students can attempt the general case in small groups, with the teacher providing assistance and hints as necessary.

Students should have time in which to practise using all of these methods in solving (and checking) quadratic equations.

9.- Students were introduced to radical equations involving one radical. A brief review of one or two exercises may be useful in introducing this topic.

Most textbooks use an algorithmic approach to solving radical equations involving two radicals, and this approach may be utilized in the classroom. Students can be given a few instructions, and then be asked to complete a few exercises, working cooperatively in small groups or individually, to solve these exercises.

After a short time, answers or possible solutions can be shared with the entire class. It should be pointed out that checking the solutions is important and the checking procedure should be carried out when solutions are presented to the class. Students should be asked to observe the solutions to determine if there are any general procedures that might be followed which make the solution to these equations simpler. Any such observations can be discussed with the entire class.

10.- Once suitable word problems have been identified, students should work cooperatively to determine the necessary information, to pose the question to be answered and to write the equation, which, when solved, will provide the solution to the problem.

Problems may be presented one or two at a time. Student answers may be shared with the entire class after students have had a few moments to work on the problems. A discussion of the main components of the problem, and the setup of the equation may help all students increase their skills in problem solving.

A few problems may be assigned to the class to complete in small groups. Some statement should be elicited from the student as to whether the problem outlined in each case is a real-world situation, and if so, who might be responsible for its solution in its real-world context.