Function: Concepts. Linear Functions. Functions in the Real World

1. Introduction to Functions.

Functions are used whenever one variable depends on another variable. This relationship between two variables is the most important thing in mathematics. It is a way of saying:

“If you tell me what x is, I can tell you what y is”.

We say that y “depends on” x, or y “is a function of” x. A few examples:

1. “The area of a circle depends on its radius.”

We know the equation for the area of a circle from primary school: 

\[ A = \pi r^2 \]

This is a function as each value of the independent variable \( r \) gives us one value of the dependent variable \( A \).

2. “The amount of money Alice makes depends on the number of days she works.”

Suppose you know that Alice makes $100 per day. Then we could make a chart like this.

<table>
<thead>
<tr>
<th>Days</th>
<th>0</th>
<th>1</th>
<th>1½</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sales</td>
<td>0</td>
<td>100</td>
<td>150</td>
<td>800</td>
</tr>
</tbody>
</table>

If Alice works this many days... ...she makes this many dollars
If you tell me how long she has worked, I will tell you how much money she has made. Her earnings “depend on” how long she works.

3. “Voltage depends on current and resistance”.

\[ V = IR \]

where \( V \) = voltage (V) \( I \) = current (A) \( R \) = resistance (Ω)

If \( I \) increases, so does the voltage (assuming resistance is constant).

If \( R \) increases, so does the voltage (assuming current is constant).

4. “Speed depends on distance travelled and time taken”

\[ s = \frac{d}{t} \]

where \( s \) = speed (m / s) \( d \) = distance (m) \( t \) = time taken (s)

If \( d \) increases, the speed goes up (assuming time is constant). If \( t \) increases, the speed goes down (assuming distance is constant).

2. Definition of a Function.

Whenever a relationship exists between two variables (or quantities) such that for every value of the first, there is only one corresponding value of the second, then we say:

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“the second variable is a function of the first variable”.

The first variable is the *independent* variable (usually $x$), and the second variable is the *dependent* variable (usually $y$).

The independent variable and the dependent variable are *real numbers*.

We use $x$ (*independent*) and $y$ (*dependent*) variables for *general cases*.

**Example 1:**

In the equation $y = 3x + 1$, $y$ is a function of $x$, since for each value of $x$, there is only one value of $y$.

If we substitute $x = 5$, we get $y = 16$ and no other value.

The values of $y$ we get depend on the values chosen for $x$. Therefore, $x$ is the *independent* variable and $y$ is the *dependent* variable.

Function notation is all about *substitution*. The value of the function $f(x)$ when $x = a$ is written as $f(a)$.

**Example 2:**

If we have $f(x) = 4x + 10$, the value of $f(x)$ for $x = 3$ is written: $f(3) = 4 \times 3 + 10 = 22$

When $x = 3$, the value of the function $f(x)$ is 22.

3. **Rectangular Coordinates.**

A good way of presenting a function is by graphical representation.

Graphs give us a visual picture of the function.

The rectangular co-ordinate system consists of:

- a) the $x$-axis
- b) the $y$-axis
- c) the origin (0,0)
- d) the four quadrants
Normally, the values of the independent variable (generally the x-values) are placed on the horizontal axis, while the values of the dependent variable (generally the y-values) are placed on the vertical axis.

The x-value, called the abscissa, is the perpendicular distance of P from the y-axis.

The y-value, called the ordinate, is the perpendicular distance of P from the x-axis.

The values of x and y together, written as (x, y) are called the co-ordinates of the point P.

Exercise 1.- Locate the points A(2, 1) and B(-4, -3) on the rectangular co-ordinate system.

Exercise 2.- Where are all points (x, y) for which x < 0 and y < 0?

Exercise 3.- Where are all the points whose abscissas equal their ordinates?

4. The Graph of a Function

The graph of a function is the set of all points whose co-ordinates (x, y) satisfy the function y = f(x). This means that for each x-value there is a corresponding y-value which is obtained when we substitute into the expression for f(x).

Since there is no limit to the possible number of points for the graph of the function, we will follow this procedure at first:

- select a few values of x
- obtain the corresponding values of the function
- plot these points joining them.

However, you are encouraged to learn the general shapes of certain common curves (like straight line, parabola, trigonometric and exponential curves) - it’s much easier than plotting points and more useful for later!

Example 1

A man who is 2 m tall throws a ball straight up and its height at time t (in s) is given by

\[ h = 2 + 9t - 4.9t^2 \quad \text{m.} \]

Graph the function.

Answer

We start at t = 0 since negative values of time have no practical meaning here.
This shape is called a parabola and is common in applications of mathematics.

**NOTE:** We could have written the function in this example as: \( h(t) = 2 + 9t - 4.9t^2 \).

### 5.- Definitions of Domain and Range.

**Domain:** The domain of a function is the complete set of possible values of the independent variable in the function.

In plain English, this definition means:

The domain of a function is the set of all possible \( x \) values which make the function work and will output real \( y \) values.

**Range:** The range of a function is the complete set of all possible resulting values of the dependent variable of a function, after we have substituted the values in the domain.

In plain English, this definition means:

The range of a function is the possible \( y \) values of a function that result when we substitute all the possible \( x \) values into the function.

### 6.- Continuous Functions.

Consider the graph of \( f(x) = x^3 - 6x^2 - x + 30 \):
We can see that there are no "gaps" in the curve. Any value of \( x \) will give us a corresponding value of \( y \). We could continue the graph in the negative and positive directions, and we would never need to take the pencil off the paper.

Such functions are called **continuous functions**.

### 7.-Linear Functions.-

What is a linear function? A linear function is one whose graph is a straight line (hence the term "linear").

How do we recognize a linear function algebraically? As follows:

A **linear function** is one that can be written in the form

\[
\begin{align*}
    f(x) &= mx + b \\
    y &= mx + b
\end{align*}
\]

where \( m \neq 0 \) and \( b \) are fixed numbers (the names \( m \) and \( b \) are traditional).

The **domain** of this function is the set of all real numbers. The **range** of \( f \) is the set of all real numbers. The graph of \( f \) is a straight line with **slope** \( m \) and **y-intercept** \( b \).

**Note:** A function \( f(x) = b \), where \( b \) is a constant real number is called a constant function. Its graph is a horizontal line at \( y = b \).

**Example 1.-** Here is a partial table of values of the linear function \( f(x) = 3x - 1 \). Fill in the missing values.

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
    x & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 \\
    y & -13 & -10 & -7 & -1 & 2 & 5 & & & \\
\end{array}
\]

Plotting a few of these points gives the following graph.

**Example 2.-** Graph the linear function \( f \) given by \( f(x) = 2x + 4 \)

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You need only two points to graph a linear function. These points may be chosen as the x and y intercepts of the graph for example.

Determine the **x-intercept**, set \( f(x) = 0 \) and solve for \( x \).

\[
2x + 4 = 0 \quad \rightarrow \quad x = -2
\]

Determine the **y-intercept**, set \( x = 0 \) to find \( f(0) \).

\[
\rightarrow \quad f(0) = 4
\]

The graph of the above function is a line passing through the points \((-2, 0)\) and \((0, 4)\) as shown below.

**Example 3.** Graph the linear function \( f \) given by \( f(x) = -(1/3)x - 1/2 \)

Determine the **x-intercept**, set \( f(x) = 0 \) and solve for \( x \).

\[
\left( -\frac{1}{3} \right)x - \frac{1}{2} = 0 \quad \rightarrow \quad -\frac{x}{3} = \frac{1}{2} \quad \rightarrow \quad x = -\frac{3}{2}
\]

Determine the **y-intercept**, set \( x = 0 \) to find \( f(0) \).

\[
\rightarrow \quad f(0) = -1/2
\]

The graph of the above function is a line passing through the points \((-3/2, 0)\) and \((0, -1/2)\) as shown below.

**Exercise 1.** Graph the linear function \( f \) given by \( f(x) = x + 3 \)

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7.1.- The Role of b in the equation \( y = mx + b \)

Let us look more closely at the above linear function, \( y = 3x - 1 \), and its graph, shown above. This linear equation has \( m = 3 \) and \( b = -1 \).

Notice that that setting \( x = 0 \) gives \( y = -1 \), the value of \( b \).

*Numerically, \( b \) is the value of \( y \) when \( x = 0 \)*

On the graph, the corresponding point \((0, -1)\) is the point where the graph crosses the \( y \)-axis, and we say that \( b = -1 \) is the *y-intercept* of the graph.

*Graphically, \( b \) is the \( y \)-intercept of the graph*

7.2.- The Role of \( m \) in the equation \( y = mx + b \)

Notice from the table that the value of \( y \) increases by \( m = 3 \) for every increase of 1 in \( x \). This is caused by the term \( 3x \) in the formula: for every increase of 1 in \( x \) we get an increase of \( 3 \times 1 = 3 \) in \( y \).

*Numerically, \( y \) increases by \( m \) units for every 1-unit increase of \( x \).*

On the graph, the value of \( y \) increases by exactly 3 for every increase of 1 in \( x \), the graph is a straight line rising by 3 for every 1 we go to the right. We say that we have a *rise* of 3 units for each *run* of 1 unit.

Similarly, we have a rise of 6 for a run of 2, a rise of 9 for a run of 3, and so on. Thus we see that \( m = 3 \) is a measure of the steepness of the line; we call \( m \) the *slope* of the line.

*Geometrically, the graph rises by \( m \) units for every 1-unit move to the right; \( m \) is the slope of the line.*

Here is a more general picture showing two "generic" lines; one with positive slope, and one with negative slope.

Mathematicians traditionally use (\( \Delta \), the Greek equivalent of the Roman letter \( D \)) to stand for "difference," or "change in." For example, we write \( \Delta x \) to stand for "the change in \( x \)."

Let us take another look at the linear equation \( y = 3x - 1 \)

Now we know that \( y \) increases by 3 for every 1-unit increase in \( x \). Similarly, \( y \) increases by \( 3 \times 2 = 6 \) for every 2-units increase in \( x \). In general, \( y \) *increases by \( 3 \times \) units for every \( \Delta x \)-units change in \( x \).*

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Using symbols, \[ \text{Slope} = m = \frac{\Delta y}{\Delta x} = 3 \]

How do these changes show up on the graph? Here again is the graph of \( y = 3x - 1 \), showing two different choices for \( \Delta x \) and the associated \( \Delta y \).

The slope of a line is given by:

\[ m = \frac{\text{Change in } y}{\text{Change in } x} = \frac{\Delta y}{\Delta x} = \frac{\text{Rise}}{\text{Run}} = \text{Slope} \]

For **positive** \( m \), the graph rises \( m \) units for every 1-unit move to the right, and **rises** \( y = m \times x \) units for every \( x \) units moved to the right.

For **negative** \( m \), the graph **drops** \( |m| \) units for every 1-unit move to the right, and drops \( |m| \times x \) units for every \( x \) units moved to the right.
8.- Real World Examples of Linear Equations Are All Around Us

I think that students understand everything better with real life examples. Many students just don’t realize that linear functions are all around them.

1.- Travel.

Let’s say we are going on a trip where we are averaging a speed of 60 miles per hour. This is a linear function.

The equation would be: \( D = 60t \), where \( D \) is the distance covered in \( t \) hours.

2.- Miles per gallon.

A car’s efficiency in terms of miles per gallon of petrol is a function.

“If a car typically gets 20 miles per gallon (mpg), and if you input 10 gallons of petrol, it will be able to travel roughly 200 miles.

The equation would be: \( D = 20g \), where \( D \) is the distance covered with \( g \) gallons.

The car’s efficiency may be a function of the car’s design, speed, temperature inside and outside of the car, and other factors.

3.- Exchange Rates.

Nowadays, “three euros are four dollars”, approximately. Therefore, 1 dollar is about 0.75 euros.

A linear equation of a function, which converts US dollars \( (D) \) to euros \( (E) \) would be:

\[ E = 0.75D \]

You would just put the number of dollars in for \( D \) and multiply by 0.75 and this will give you the number of euros you would get for your US dollars.
Similarly, “one pound (GBP) is about 1.12 euros”. The linear equation that converts GB Pounds \((P)\) to euros \((E)\) would be: \[ E = 1.12P \]

These are real world examples of a linear function. As you can see, you don’t have to look so far for real life examples of linear functions.

4.-Exercise.-

The fixed cost for a company to operate a certain plant is $3,000 per day. It also costs $4 for each unit produced in the plant. Express the daily cost \(C\) of operating the plant as a function of the number \(n\) of units produced.

9.-REAL LIFE GRAPHS.-

**Exercise 1.** The graph below shows the cost of a phone call depending on the length of the phone call in minutes.

Use the graph to work out:

a) the cost of a 5 minute phone call

…………………………

b) how long you can talk for £5

…………………………

c) Why does the graph not start at the origin?

…………………………………………………………………………………………………………………………………………………

d) Write down a formula connecting the cost \((C)\) of a phone call and the length of the phone call in minutes \((m)\).

………………………………………………………………………………………………………………………………………………………………………………
Exercise 2.-

Hannah’s journey to the park is shown below. She calls for her friend, Emma, on the way to the park.

![Graph showing distance from home (m) over time (mins)]

a) Hannah left her house at 10.25 a.m. At what time did she arrive at Emma’s house?

b) How long did they spend at the park?

c) The graph clearly shows that the journey home from the park is downhill. Do you agree with this statement? Give reasons for your answer.

Exercise 3.- The graph shows the motion of a high-speed model car.

![Graph showing distance (m) over time (s)]

a) How far away from its starting point was the car after 7 seconds? Answer: .............................................m.

b) How far did the car travel in the first 4 seconds? Answer: .............................................m.

c) How fast did the car go in the first 4 seconds? Answer: ..............................................................m/s.

d) Between what times was the car travelling fastest? Answer: ....................................................

e) Describe what is happening between 4 and 6 seconds. Answer: ................................................................